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Lévy-Process Intrinsic Statistical Solutions of a Randomly Forced Burgers Equation

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Abstract We consider Burgers equation forced by a brownian in space and white noise in time process $\partial_t u + \frac{1}{2}\partial_x(u)^2 = f(x,t)$, with $E(f(x,t)f(y,s)) = \frac{1}{2}(|x| + |y| - |x-y|) \times \delta(t-s)$ and we show that there exist intrinsic statistical solutions that are Lévy processes at any given positive time. We give the evolution equation for the characteristic exponent of such solutions; in particular we give the explicit solution in the case $u_0(x) = 0$.

Keywords Burgers · Random forcing · Lévy process · Turbulence

1 Introduction

We consider the randomly forced Burgers equation $(*)$ $\partial_t u + \frac{1}{2}\partial_x(u^2) = f(x,t)$, where f satisfies $E(f(x,t)f(y,s)) = A(x,y)\delta(t-s)$ with $A(x,y) = \frac{1}{2}(|x| + |y| - |x-y|)$: f is a brownian in space/ white noise in time process (the brownian is considered on the whole space, and is obtained with two independent Brownians, one on the left and one on the right). Burgers equation has originally been introduced as a $1D$ model of turbulence [3]. It is certainly too crude, but it is still a good idea when trying to find new approaches to the "real" 3D turbulence as given by Euler equation, to start by looking at Burgers. In this point of view, the randomly forced Burgers equation is a crude simplification on the way to the description of forced turbulence. A variant of our problem was considered by [6], where they considered a Gaussian in space/white noise in time forcing, satisfying $E(f(x,t)f(y,s)) = A(x-y)\delta(t-s)$ and obtained all the hierarchy of n-point densities evolution equations. We will proceed quite differently, basing our study on the result shown

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by Carraro and Duchon in [4], and by Bertoin in [2], that there are statistical solutions (in a sense that we will recall later) of the Burgers equation without any forcing, which at any given time are Lévy processes without positive jumps. This is actually a particular case of the fact that the Burgers equation conserves the Markovian (in space) property of processes [5]. We will show here in the same spirit that the brownian in space/white noise in time forcing allows to keep the Lévy property: there are intrinsic statistical solutions that are at any given time spatial Lévy processes without positive jumps. We will first derive an evolution equation for the characteristic exponent of such a Lévy solution; then we will prove that this equation has solutions that are for all time $t > 0$ characteristic exponents of a Lévy process without positive jumps. Hence there are indeed such statistical solutions. Moreover, our method allows us to write explicitly for any positive time the exponent of the Lévy process that one gets when starting from $u_0(x) = 0$. Even if no uniqueness theorem has been proven for intrinsic statistical solutions, the fact that these Lévy solutions have no positive jumps shows that they should be meaningful physically.

2 Statistical Solutions

We will closely follow [4] (see also [9]). Let E be the space of càdlàg real functions. We will call $\mathcal{C}(E)$ the smallest σ-algebra such that for each $x \in \mathbb{R}$, $u \mapsto u(x)$ is measurable, and $\mathcal{C}'(E)$ the smallest σ-algebra such that for each $(x, y) \in \mathbb{R} \times \mathbb{R}$, $u \mapsto u(x) - u(y)$ is measurable. Let $\mathcal{D}_0$ be the set of real C^∞ functions v with compact support such that $\int_{\mathbb{R}} v(x)dx = 0$. A probability μ on $(E, \mathcal{C}(E))$ is then characterized by its characteristic function

$$v \in \mathcal{D} \mapsto \hat{\mu}(v) := \int_E \exp\left[i \int_{\mathbb{R}} u(x)v(x)dx\right] d\mu(u)$$

whereas a probability μ on $(E, \mathcal{C}'E))$ is characterized by its characteristic function $v \in \mathcal{D}_0 \mapsto \hat{\mu}(v)$.

Let $u_0 : (\Omega, \mathcal{A}, P) \to E$ be a random process, defined on some probability space, and let $\mu_0 : \mathcal{C}(E) \to [0, 1]$ denote its probability law: u_0 will be our initial condition. We will let u evolve according to the non forced Burgers equation, obtaining thus a family of processes u_t where $u_t(x) = u(x, t)$ is the solution of Burgers at time t. Then it is not difficult to check that the characteristic function $\hat{\mu}_t$ of u_t verifies:

$$\begin{aligned} \partial_t \hat{\mu}_t(v) &= \partial_t \left(\int_E \exp\left[i \int_{\mathbb{R}} u(x)v(x)dx\right] d\mu_t(u) \right) \\ &= \int_E \partial_t \left\{ \exp\left[i \int_{\mathbb{R}} u(x,t)v(x)dx\right]\right\} d\mu_0(u_0) \\ &= \int_E \exp\left[i \int_{\mathbb{R}} u(x,t)v(x)dx\right] \left(\partial_t \left[i \int_{\mathbb{R}} u(x,t)v(x)dx\right]\right) d\mu_0(u_0) \\ &= \int_E \exp\left[i \int_{\mathbb{R}} u(x)v(x)dx\right] \left[i \int_{\mathbb{R}} \frac{1}{2} u(x)^2 v'(x)dx\right] d\mu_t(u) \end{aligned} \tag{1}$$

This motivated the definition given in [4] of a statistical solution of Burgers equation, as a family of random processes u_t whose characteristic functions are solutions of (1).

Here we will take for u_0 a Lévy process without positive jumps, that is a càdlàg process with stationary homogeneous increments and without positive jumps, and we are really interested only in the law of the increments. Hence u_0 really defines a probability law on

$(E, \mathcal{C}'E)$). But $u \mapsto \int_{\mathbb{R}} u(x)^2 v'(x)dx$ is not $C'(E)$ measurable, hence this evolving equation for $\hat{\mu}$ makes no sense. One way around this is to take a different approach and to work with the Hopf-Cole construction as in [2] or [7]. Another way was explored in [4]: it is based on the notion of *intrinsic statistical solution*, and involves working with $u(x,t) - \frac{1}{b-a}\int_a^b u(x,t)dx$ when $b-a$ gets large:

Definition 1 An intrinsic statistical solution of Burgers equation is a set $(\mu_t)_{t\geq 0}$ of probabilities on $(E, \mathcal{C}'(E))$ such that for any $v \in \mathcal{D}_0$,

$$\partial_t \hat{\mu}_t(v) = \frac{i}{2} \lim_{a\to-\infty, b\to+\infty} \int_E \left(\int_{\mathbb{R}} \left(u(x) - \frac{1}{b-a} \int_a^b u(y)dy \right)^2 v'(x)dx \right) \times \exp\left[i \int_{\mathbb{R}} uv \right] d\mu_t(u)$$

A justification for this definition is the proposition, proven in [4], that if u is a homogeneous process of finite variance, then

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_E \left(\int_{\mathbb{R}} \frac{i}{2} \left(u(x) - \frac{1}{b-a} \int_a^b u(y)dy \right)^2 v'(x)dx \right) \exp\left[i \int_{\mathbb{R}} uv \right] d\mu(u) = \int_E \left(\int_{\mathbb{R}} i\frac{1}{2} u(x)^2 v'(x)dx \right) \exp\left[i \int_{\mathbb{R}} uv \right] d\mu(u)$$

This notion of intrinsic statistical solution has the drawback that there is up to now no general proof for the existence of a unique solution with no positive jumps, nor that this solution coincides with the Hopf-Cole solution. Yet it allows to get explicit results, and one can check that the solution found has no positive jumps. It is then reasonable to think that such a solution should be physically meaningful. As a matter of fact, in the no forcing case, the statistical solution thus found in [4] for a brownian initial condition does indeed coincide with the solution found in [2] by Hopf-Cole method.

In order to work with Lévy processes we will need their characterization by means of their exponents.

Definition 2 If u is a Lévy process, its characteristic exponent ψ is defined by $\forall x < y$, $\forall w \in \mathbb{R}$,

$$E\{\exp i[w(u(y) - u(x))]\} = \exp[(y-x)\psi(w)] \tag{2}$$

If the process is of finite variance without positive jumps, one can use analytic continuation to define its Laplace exponent $\phi(w) := \psi(-iw)$ as a function on $\mathbb{R}^+$.

In [4] it is shown that if u_0 is a Lévy process of finite variance such that $\phi_0'(0) \geq 0$, and if we let u evolve according to the non forced Burgers equation, u_t is still a Lévy process; its characteristic exponent ψ_t satisfies the evolution equation $\partial_t \psi = i\partial_w(\psi^2)/2$.

Let now f be the random forcing, independent on u_0. We will denote by μ_F its probability law. Restricted to events involving a fixed time, μ_F is the law of a brownian motion defined on $\mathbb{R}$ by gluing one brownian on $\mathbb{R}^+$ and the reflection of another independent brownian on $\mathbb{R}^-$. Such a brownian on $\mathbb{R}$ is a continuous process with independent homogeneous increments null at 0.

The basic idea now is the following: if u_t is a Lévy process with probability μ_t on $C'(E)$, and if we let u_t evolve according to our forced Burgers equation, $u(x, t+dt)$ will be the sum of $u_{NF}(x, t+dt)$ and $\sqrt{dt}B(x)$, where u_{NF} is the process we would have obtained without any forcing, and $B(x)$ is a brownian motion independent of u_{NF}. $u(x, t+dt)$ should thus still be a process with homogeneous independent increments, that is, a Lévy process with exponent the sum of the exponents. Since the characteristic exponent of a brownian is $\frac{x^2}{2}$, and since the evolution equation for the characteristic exponent in case of no forcing is $\partial_t\psi(t, w) = i\partial_w\psi(t, w)\psi(t, w)$, the evolution equation of the characteristic exponent of μ with random forcing should be $\partial_t\psi(t, w) = i\partial_w(\psi^2)/2 + \frac{w^2}{2}$.

One can also get this result in a more rigorous way, yet still formal, by working with characteristic functions. Let us first define statistical solutions for the forced Burgers equation. Let $u(x, t)$ be a (weak) solution of our forced Burgers equation with $u(., 0) = u_0$, $u(., t) \in E$ for $t > 0$. Let μ_t denote the law of $u(., t)$ on $\mathcal{C}(E)$. If v in $\mathcal{D}$, one needs to compute

$$\partial_t\hat{\mu}_t(v) = \int_F\int_E \partial_t\left\{\exp\left[i\int_{\mathbb{R}} u(x, t)v(x)dx\right]\right\}d\mu_0(u_0)d\mu_F$$

If X_t is a continuous semi-martingale, Ito's formula (see [8]) states that

$$d(\exp(iX_t)) = i\exp(iX_t)dX_t - \frac{1}{2}\exp(iX_t)d\langle X, X\rangle_t$$

Applying it formally to $X_t = \int_{\mathbb{R}} u(x, t)v(x)dx$, which satisfies $\dot{X}_t = \int_{\mathbb{R}} u^2(x, t)v'(x)dx + \int_{\mathbb{R}} v(x)f(x, t)dx$, yields

$$\begin{aligned}
&\partial_t\left\{\exp\left[i\int_{\mathbb{R}} u(x, t)v(x)dx\right]\right\}\\
&\quad= \exp\left[i\int_{\mathbb{R}} u(x, t)v(x)dx\right]\left(i\int_{\mathbb{R}} u(x, t)^2 v'(x)dx\right.\\
&\qquad\left. + i\int_{\mathbb{R}} f(x, t)v(x)dx - \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)\partial_t\langle u(x, t)u(y, t)\rangle_t dxdy\right)
\end{aligned}$$

Hence one finally gets

$$\begin{aligned}
\partial_t\hat{\mu}_t(v) = \int_E &\left(\int_{\mathbb{R}} i\frac{1}{2}u(x)^2v'(x)dx - \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x, y)dxdy\right)\\
&\times\exp\left[i\int_{\mathbb{R}} uv\right]d\mu_t(u)
\end{aligned}$$

Hence our definitions:

Definition 3 A statistical solution of the forced Burgers equation (*) is a set $(\mu_t)_{t\geq 0}$ of probabilities on $(E, \mathcal{C}(E))$ such that for any $v \in \mathcal{D}$,

$$\begin{aligned}
\partial_t\hat{\mu}_t(v) = &\int_E\left(\int_{\mathbb{R}} i\frac{1}{2}u(x)^2v'(x)dx\right)\exp\left[i\int_{\mathbb{R}} uv\right]d\mu_t(u)\\
&- \left(\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x, y)dxdy\right)\hat{\mu}_t(v)
\end{aligned}$$

Definition 4 An intrinsic statistical solution of the forced Burgers equation (*) is a set $(\mu_t)_{t\geq 0}$ of probabilities on $(E, \mathcal{C}'(E))$ such that for any $v \in \mathcal{D}_0$,

$$\partial_t \hat{\mu}_t(v) = \lim_{a\to-\infty, b\to+\infty} \int_E \left(\int_{\mathbb{R}} \frac{i}{2} \left(u(x) - \frac{1}{b-a} \int_a^b u(y)dy \right)^2 v'(x)dx \right) \times \exp\left[i \int_{\mathbb{R}} uv \right] d\mu_t(u) - \left(\frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x,y)dxdy \right) \hat{\mu}_t(v) \quad (3)$$

One can remark that for $A(x,y) = \frac{1}{2}(|x| + |y| - |x-y|)$ and $v \in \mathcal{D}$, $\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y) \times A(x,y)dxdy = \int_0^{+\infty}(w(x)^2 + (w(-\infty) - w(-x))^2)dx$ where $w(x) = \int_x^{+\infty} v(y)dy$. If moreover $v \in \mathcal{D}_0$, $\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x,y)dxdy = \int_{\mathbb{R}} w(x)^2 dx$.

3 Lévy Solutions

3.1 Evolution Equation

Now we can look for solutions that would be Lévy processes with finite variance. Such processes are characterized by their exponent ψ_t, or by their Laplace exponent ϕ_t.

Let us recall here a useful lemma characterizing Laplace exponents of Lévy processes.

Lemma 1 *A function $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$ is the Laplace exponent of a homogeneous Lévy process with finite variance and without positive jumps if and only if ϕ is C^∞ on $]0, +\infty[$, C^2 on $\mathbb{R}^+$, $\phi(0) = 0$ and ϕ'' is completely monotonous.*

Moreover, Bernstein's lemma asserts that a function g continuous on $\mathbb{R}^+$, C^∞ on $]0, +\infty[$ is completely monotonous if and only if $\forall n \in \mathbb{N}$, $(-1)^n g^{(n)} \geq 0$. Injecting the definition of the exponents (2) in the evolution equation, one gets the following result.

Theorem 1 *Let $(\psi_t)_{t\geq 0}$ be a family of exponents of Lévy processes of finite variance, such that $\forall w \in \mathbb{R}$ $t \mapsto \psi_t(w)$ is differentiable and $\partial_t \psi_t$ is locally bounded. Then ψ_t characterizes an intrinsic statistical solution of our forced Burgers equation if and only if it satisfies the equation*:

$$\forall w \in \mathbb{R}, \quad \partial_t \psi(t,w) = i\partial_w \psi(t,w)\psi(t,w) - \frac{w^2}{2}$$

If the family of exponents is without positive jumps, the equation for ϕ is

$$\partial_t \phi(t,w) = -\partial_w \phi(t,w)\phi(t,w) + \frac{w^2}{2} \quad (4)$$

Proof Suppose that u_t is a family of Lévy processes with exponents $(t,w) \mapsto \psi(t,w)$. If v is a function in $\mathcal{D}_0$, the definition of ψ gives

$$\hat{\mu}_t(v) = \exp\left(\int_{\mathbb{R}} \psi\left(t, \int_x^{+\infty} v(y)dy\right) dx \right)$$

(as can be seen by considering first simple functions). Hence, thanks to local boundedness,

$$\partial_t \hat{\mu}_t(v) = \hat{\mu}_t(v) \int_{\mathbb{R}} \partial_t \psi\left(t, \int_x^{+\infty} v(y)dy\right)dx = \hat{\mu}_t(v) \int_{\mathbb{R}} \partial_t \psi(t, w(x))dx$$

where we denote as before $w(x) = \int_x^{+\infty} v(y)dy$. The right hand side of (3) is more complicated to get. One needs to compute

$$\lim_{a\to-\infty, b\to+\infty} \int_E \left(\int_{\mathbb{R}} i\frac{1}{2}\left(u(x) - \frac{1}{b-a}\int_a^b u(y)dy\right)^2 v'(x)dx\right) \exp\left[i\int_{\mathbb{R}} uv\right] d\mu_t(u)$$

in terms of ψ. This was done in [4] where it is shown that this limit is

$$i\left(\int_{\mathbb{R}} \partial_w \psi(t, w(x))\psi(t, w(x))dx\right)\hat{\mu}_t(v)$$

The other term involves

$$\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x, y)dxdy$$

We have to write it in terms of w: as noticed before, the specific form of A yields

$$\iint_{\mathbb{R}\times\mathbb{R}} v(x)v(y)A(x, y)dxdy = \int_{\mathbb{R}} w^2(x)dx$$

Finally, we must have for all v in $\mathcal{D}_0$,

$$\int_{\mathbb{R}} \partial_t \psi(t, w(x))dx = i\int_{\mathbb{R}} \partial_w \psi(t, w(x))\psi(t, w(x))dx - \frac{1}{2}\int_{\mathbb{R}} w^2(x)dx$$

This is true if and only if

$$\forall w \in \mathbb{R}, \quad \partial_t \psi(t, w) = i\partial_w \psi(t, w)\psi(t, w) - \frac{w^2}{2}$$

One should note here that the particular form of A is crucial: otherwise one would be unable to get rid of the integral signs. □

The stationary solution of (4) is $\phi(w) = \frac{w^{\frac{3}{2}}}{\sqrt{3}}$: this is the exponent of a Lévy-stable process (which of course is not of finite variance: ϕ is not C^2 in 0). Speaking informally, this is an "invariant measure" of our forced Burgers equation.

Of course this theorem does not guarantee the existence of such solutions. This will be the point of our next result. In order to do this, we will have to take a close look at (4).

Equation (4) can be solved using characteristics. Characteristics for (4) are curves $(\gamma_{x_0}(s), \lambda_{x_0}(s))$ that are solutions of the system

$$\begin{cases} \frac{d\gamma}{ds} = \lambda(s) \\ \frac{d\lambda}{ds} = \frac{\gamma(s)^2}{2} \end{cases} \tag{5}$$

with initial condition $\gamma_{x_0}(0) = x_0$, $\lambda_{x_0}(0) = \phi_0(x_0)$. Each x_0 corresponds to a different characteristics. To find the value of $\phi(x, t)$, one has to find x_0, if it exists, such that $\gamma_{x_0}(t) = x$; if x_0 exists and is unique, then $\phi(x, t) = \lambda_{x_0}(t)$.

One can notice that system (5) describes the motion of a particle in a potential $V(x) = -\frac{x^3}{6}$, starting at x_0 with a velocity $\phi_0(x_0)$. Thanks to energy conservation, it is thus equivalent to $\lambda = \gamma'$, $\gamma^3 - 3(\gamma')^2 = x_0^3 - 3\phi_0(x_0)^2$, $\gamma(0) = x_0$.

These remarks, and the preceding lemma characterizing Laplace exponents, are the basic ingredients of the next result.

Theorem 2 *Let ϕ_0 be the exponent of a Lévy process of finite variance without positive jumps such that $\phi_0'(0) \geq 0$. Then (4) with initial condition ϕ_0 admits a unique solution which is for all $t > 0$ the exponent of a homogeneous Lévy process with finite variance and without positive jumps. Such a Lévy process is thus an intrinsic statistical solution of the randomly forced Burgers equation.*

If $\phi_0 = 0$, the solution is $\phi(x,t) = t^{-3}\sqrt{(xt^2)^3 - F(xt^2)^6}$ where F is the inverse of $x \mapsto 4^{\frac{1}{3}} x^2 \mathcal{P}(\omega_2 + \frac{x}{\sqrt{3}4^{\frac{1}{3}}})$. Here $\mathcal{P}$ is the Weierstrass function and ω_2 its half-period.

As mentioned before, there is no uniqueness proven for intrinsic statistical solutions. However the fact that jumps stay negative, is a strong sign that these solutions we exhibit should be "real" ones. Moreover here as far as we know no other method can provide us with such explicit formulae.

Proof We will start the proof by looking at the particular case $\phi_0 = 0$. It is instructive to see how things go, and it is anyway necessary to deal with it separately. The general case will be covered afterwards.

3.2 The Particular Case $\phi_0 = 0$

In this particular case (corresponding to u_0 constant), the system (5) is equivalent to $\lambda = \gamma'$, $\gamma^3 - 3(\gamma')^2 = x_0^3$, $\gamma(0) = x_0$. The solution of this equation is $\gamma(s) = 4^{\frac{1}{3}} x_0 \mathcal{P}(\omega_2 + \frac{\sqrt{x_0}}{\sqrt{3}4^{\frac{1}{3}}} s)$ where $\mathcal{P}$ is the Weierstrass function solution of $y'^2 - 4y^3 + 1 = 0$ (with invariants $g_2 = 0$ and $g_3 = 1$), and $\omega_2 = \frac{\Gamma^3(\frac{1}{3})}{4\pi}$ is the half period of $\mathcal{P}$, satisfying $\mathcal{P}'(\omega_2) = 0$. γ is 1–1 from $[0, \frac{\sqrt{3}4^{\frac{1}{3}}}{\sqrt{x_0}}\omega_2[$ to $[x_0, +\infty[$ (see [1]); it is C^∞ and strictly increasing on $]0, \frac{\sqrt{3}4^{\frac{1}{3}}}{\sqrt{x_0}}\omega_2[$. In order to find $\phi(x,t)$, one needs to find x_0 such that $4^{\frac{1}{3}} x_0 \mathcal{P}(\omega_2 + \frac{\sqrt{x_0}}{\sqrt{3}4^{\frac{1}{3}}} t) = x$. Hence $x_0 = \frac{F(xt^2)^2}{t^2}$ where F is the inverse of $x \mapsto 4^{\frac{1}{3}} x^2 \mathcal{P}(\omega_2 + \frac{x}{\sqrt{3}4^{\frac{1}{3}}})$. F is continuous from $[0, +\infty[$ to $[0, \sqrt{4}4^{\frac{1}{3}}\omega_2[$ and C^∞ on $]0, +\infty[$. Near 0, one has $4^{\frac{1}{3}} x^2 \mathcal{P}(\omega_2 + \frac{x}{\sqrt{3}4^{\frac{1}{3}}}) = x^2 + \frac{x^4}{4} + O(x^6)$, hence $F(x) = \sqrt{x} - \frac{x^{\frac{3}{2}}}{8} + o(x^2)$.

Thus the solution of (4) with initial condition $\phi_0 = 0$ is $\phi(x,t) = t^{-3}\sqrt{(xt^2)^3 - F(xt^2)^6}$; the development of F near 0 ensures that $x \mapsto \phi(x,t)$ is C^2 on $[0, +\infty[$. The proof that ϕ'' is completely monotone will be done for the general case.

3.3 General Case

As a preliminary remark, since ϕ_0'' is completely monotone, $\phi_0' \geq 0$ on $\mathbb{R}^+$, hence ϕ_0 is nonnegative and increasing on $\mathbb{R}^+$. Moreover, if there is an interval $I = [0, a[$ such that $\phi_0(w) = 0\ \forall w \in I$, then $\phi_0'' = 0$ on I. But ϕ'' is decreasing (because $\phi_0^{(3)} \leq 0$) hence $\phi_0'' = 0$

on $\mathbb{R}^+$. Therefore $\phi_0 = 0$ on $\mathbb{R}^+$ or $\phi_0 > 0$ on $]0, +\infty[$. The particular case $\phi_0 = 0$ has been covered in the preceding section (except for complete monotonocity), hence we will assume here that $\phi_0 > 0$ on $\mathbb{R}^{+*}$.

Everywhere in this proof $\phi(x,t)'$,$\phi(x,t)''$, $\phi(x,t)^{(n)}$ will refer to partial derivatives with respect to x.

As noticed before, the solution $\gamma_{x_0}(t)$ of the system (5) is the trajectory of a particle in a potential $V(x) = -\frac{x^3}{6}$, starting at x_0 with a velocity $\phi_0(x_0)$. Hence $\gamma_{x_0}(t)$ is a continuous increasing function of x_0. More precisely, if $x_0 = 0$ $\gamma_0(t) = 0$, else if $x_0 \neq 0$,

$$\gamma_{x_0}(t) = x \quad \Leftrightarrow \quad t = \int_{x_0}^{x} \frac{dy}{\sqrt{y^3 - x_0^3 + 3\phi_0^2(x_0)}}$$

The function $h : \{(x, x_0)/0 < x_0 \le x\} \to \mathbb{R}^+$, defined by $h(x, x_0) = \int_{x_0}^{x} \frac{dy}{\sqrt{y^3 - x_0^3 + 3\phi_0^2(x_0)}}$ is C^∞ ($\phi_0(x_0) > 0$, therefore this integral is well defined); $x_0 \mapsto h(x, x_0)$ is a diffeomorphism from $]0, x]$ into $\mathbb{R}^+$ hence for all $t > 0$ $\gamma_{x_0}(t) = x$ defines uniquely x_0 as a C^∞ function of (x,t) on $]0, +\infty[^2$, satisfying $\lim_{x\to 0} x_0(x,t) = 0$.

Hence (4) admits as a solution for all t, $\phi(x,t) = \sqrt{x^3 - x_0(x,t)^3 + 3\phi_0^2(x_0)}$, $\phi(0,t) = 0$. $x \to \phi(x,t)$ is C^∞ on $]0, +\infty[$. We still have to check that ϕ'' is completely monotone and that ϕ is C^2 at 0.

Let us prove that ϕ'' is completely monotone. This proof will also work for $\phi_0 = 0$. The keypoint is to notice that $\frac{d}{dt}(\phi(\gamma_{x_0}(t), t)) = \lambda_{x_0}(t)\phi'(\gamma_{x_0}(t), t) + \frac{\partial \phi}{\partial t}(\gamma_{x_0}(t), t) = \phi\phi'(\gamma_{x_0}(t), t) + \frac{\partial \phi}{\partial t}(\gamma_{x_0}(t), t)$. Hence by deriving twice equation (4) with respect to w, and by setting $w = \gamma_{x_0}(t)$ one gets

$$\frac{d}{dt}(\phi''(\gamma_{x_0}(t), t)) + 3\phi'\phi''(\gamma_{x_0}(t), t) = 1$$

Thus, for every x_0, $t \mapsto \phi''(\gamma_{x_0}(t), t))$ satisfies a differential inequality of the form $\frac{du}{dt} + 3fu \ge 0$ (where $f = \phi'(\gamma_{x_0}(t), t)$). Since $\phi_0'' \ge 0$, we deduce $\phi''(\gamma_{x_0}(t), t) \ge 0$ for all $t > 0$. Since it is true for all x_0, $\phi''(x,t) \ge 0$ for all $x > 0$, $t > 0$.

One can now proceed by induction. Suppose $(-1)^n \phi^{(n)}(x,t) \ge 0$ for all $2 \le n \le N$. Then by deriving (4) $N+1$ times one gets

$$\frac{d}{dt}(\phi^{(N+1)}(\gamma_{x_0}(t), t)) + (N+2) f \phi^{(N+1)}(\gamma_{x_0}(t), t))$$

$$= -\sum_{k=2}^{N} \binom{N+1}{k} \phi^{(k)}(\gamma_{x_0}(t), t))\phi^{(N+2-k)}(\gamma_{x_0}(t), t))$$

Now the induction hypothesis guarantees that each term in the sum on the right hand side of this equation is of the sign of $(-1)^k(-1)^{N+2-k}$. Hence the right hand side is of sign $(-1)^{N+1}$. Thus $(-1)^{N+1}\phi^{(N+1)}(\gamma_{x_0}(t), t)$ verifies $\frac{du}{dt} + (N+2) f u \ge 0$. One concludes as before that $(-1)^{N+1}\phi^{(N+1)}(x,t) \ge 0$.

Hence $\phi(x,t)$ is completely monotone.

Let us now prove that ϕ is C^2 at $x = 0$: we need to know how x_0 behaves near $x = 0$. In order to do this we will consider the solution of $y^3 - 3y'^2 + C(x_0) = 0$, $y(0) = x_0$, where $C(x_0) = 3\phi_0(x_0)^2 - x_0^3$. Its explicit expression depends on the sign of C. We know that $\lim_{x\to 0} x_0 = 0$. Hence there are really two cases that have to be dealt with: $\phi_0'(0) > 0$ and $\phi_0'(0) = 0$.

– *If* $\phi_0'(0) = 0$

ϕ_0 is C^2 at 0, hence $\phi_0(x_0) = k^2 x_0^2 + o(x_0^2)$ where k is a constant, and $C(x_0) = -x_0^3 + 3k^4 x_0^4 + o(x_0^4)$ is negative when x_0 is small enough. Hence $\gamma_{x_0}(t) = A\mathcal{P}(bt + c)$ where $A^3 = -4C(x_0)$, $b = \sqrt{\frac{A}{12}}$, $\mathcal{P}$ is as before the Weierstrass function with invariants $g_2 = 0$, $g_3 = 1$, and c verifies $A\mathcal{P}(c) = x_0$.

From the expansion $\mathcal{P}(\omega_2 + \epsilon) = 4^{-\frac{1}{3}} + 3.4^{-\frac{2}{3}}\epsilon^2 + o(\epsilon^2)$ we deduce

$$A = 4^{\frac{1}{3}} x_0 (1 - k^4 x_0) + o(x_0^2)$$

$$b = \sqrt{\frac{4^{\frac{1}{3}}}{12}} \sqrt{x_0} + o(\sqrt{x_0})$$

$$c = \omega_2 + \sqrt{\frac{4^{\frac{1}{3}}}{3}} k^2 \sqrt{x_0} + o(\sqrt{x_0})$$

$$\gamma_{x_0}(t) = x_0 + \left(k^2 t + \frac{t^2}{4}\right) x_0^2 + o(x_0^2)$$

$$x_0(x,t) = x - \left(k^2 t + \frac{t^2}{4}\right) x^2 + o(x^2)$$

$$\phi(x,t) = \sqrt{\frac{x^3 - x_0^3 + 3\phi_0(x_0)^2}{3}} = \left(k^2 + \frac{t}{2}\right) x^2 + o(x^2)$$

Hence ϕ is C^2 at $x = 0$.

– *If* $\phi_0'(0) = a > 0$

Now $C(x_0) = 3a^2 x_0^2 + k x_0^3 + o(x_0^3)$, where k is a constant, is positive when x_0 is small enough. Hence $\gamma_{x_0}(t) = A\mathcal{Q}(bt + c)$ where $A^3 = 4C(x_0)$, $b = \sqrt{\frac{A}{12}}$, and $\mathcal{Q}$ is now the Weierstrass function with invariants $g_2 = 0$, $g_3 = -1$; and c must verify $A\mathcal{Q}(c) = x_0$.

$\mathcal{Q}$ is related to $\mathcal{P}$ by $\mathcal{Q}(z) = -\mathcal{P}(iz)$ for all z in $\mathbb{C}$. $\mathcal{Q}$ admits a real zero z_1, with $\mathcal{Q}'(z_1) = 1$ and $\mathcal{Q}''(z_1) = \mathcal{Q}'''(z_1) = 0$. It is not difficult to get the expansion $\mathcal{Q}(z_1 + \epsilon) = \epsilon + \frac{\epsilon^4}{2} + o(\epsilon^4)$, and:

$$A = (12a^2)^{\frac{1}{3}} x_0^{\frac{2}{3}} \left(1 + \frac{k}{9a^2} x_0 + o(x_0)\right)$$

$$b = \left(\frac{a x_0}{12}\right)^{\frac{1}{3}} \left(1 + \frac{k}{18a^2} x_0 + o(x_0)\right)$$

$$c = z_1 + \left(\frac{x_0}{12a^2}\right)^{\frac{1}{3}} - a^{-\frac{8}{3}} \left(\frac{12^{-\frac{4}{3}}}{2} + k\frac{12^{-\frac{1}{3}}}{9}\right) x_0^{\frac{4}{3}} + o(x_0^{\frac{4}{3}})$$

$$\gamma_{x_0}(t) = x_0(1 + at) + \left(\frac{k}{18a} t + \frac{1}{24a^2}((1 + at)^4 - 1)\right) x_0^2 + o(x_0^2)$$

$$x_0(x,t) = \frac{x}{1 + at} - \left(\frac{k}{18a} t + \frac{1}{24a^2}((1 + at)^4 - 1)\right) \frac{x^2}{(1 + at)^3} + o(x^2)$$

$$\phi(x,t) = \sqrt{\frac{x^3 - x_0^3 + 3\phi_0(x_0)^2}{3}} = \frac{ax}{1+at}\sqrt{1 + K_2 x + o(x)}$$

where

$$K_2 = \frac{1 + 3(1+at)^4 + 4k(1 + \frac{2at}{3})}{12a^2(1+at)^2}$$

Hence ϕ is here also C^2 at $x = 0$. □

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